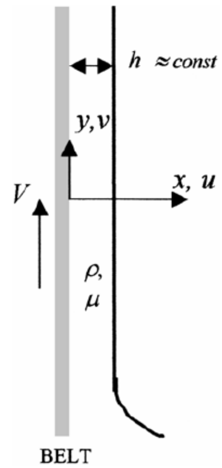
1) A belt moves upward at velocity V, dragging a film of viscous liquid of thickness h, as shown in the figure. Near the belt, the film moves upward due to no-slip. At its outer edge, the film moves downward due to gravity. Assuming that the only nonzero velocity is $v(x)$, with zero shear stress at the outer film edge, derive a formula for (a) $v(x)$; (b) the average velocity V_{avg} in the film; and (c) the wall velocity V_c for which there is no net flow either up or down.

(30)



Solution: (a) The assumption of parallel flow, $u = w = 0$ and $v = v(x)$, satisfies continuity and makes the x- and z-momentum equations irrelevant. We are left with the y-momentum equation: 4

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} - \rho g + \mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$ 7

There is no convective acceleration, and the pressure gradient is negligible due to the free surface. We are left with a second-order linear differential equation for $v(x)$:

$$\frac{d^2v}{dx^2} = \frac{\rho g}{\mu} \quad \text{Integrate: } \frac{dv}{dx} = \frac{\rho g}{\mu}x + C_1 \quad \text{Integrate again: } v = \frac{\rho g}{\mu}\frac{x^2}{2} + C_1 x + C_2$$ 2

At the free surface, $x = h$, $\tau = \mu(dv/dx) = 0$, hence $C_1 = -\rho gh/\mu$. At the wall, $v = V = C_2$. The solution is

5 1 5 1

$$v = V - \frac{\rho gh}{\mu}x + \frac{\rho g}{2\mu}x^2 \quad \textit{Ans. (a)}$$ 1

(b) The average velocity is found by integrating the distribution $v(x)$ across the film:

$$v_{avg} = \frac{1}{h}\int_0^h v(x)\,dx = \frac{1}{h}\left[Vx - \frac{\rho ghx^2}{2\mu} + \frac{\rho gx^3}{6\mu}\right]_0^h = V - \frac{\rho gh^2}{3\mu} \quad \textit{Ans. (b)}$$ 1

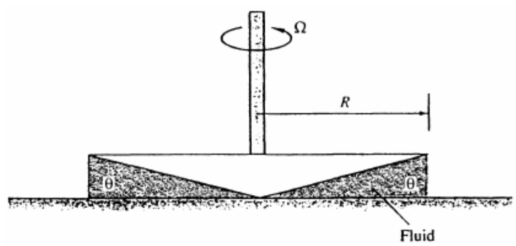
2

(c) Since $hv_{avg} \equiv Q$ per unit depth into the paper, there is no net up-or-down flow when

$$V = \rho gh^2/(3\mu) \quad \textit{Ans. (c)}$$ 1

2) The torque *M* required turning the cone-plate viscometer in figure below depends upon the radius *R*, rotation rate Ω, fluid viscosity μ, and cone angle θ. Rewrite this relation in dimensionless form. How does the relation simplify if it is known that *M* is proportional to θ?

(30)



Solution: Establish the variables and their dimensions:

$$\underset{\{ML^2/T^2\}}{M} = \text{fcn}(\underset{\{L\}}{R}, \underset{\{1/T\}}{\Omega}, \underset{\{M/LT\}}{\mu}, \underset{\{1\}}{\theta}) \quad \mathbf{10}$$

Then $n = 5$ and $j = 3$, hence we expect $n - j = 5 - 3 = \mathbf{2}$ Pi groups, capable of only one reasonable arrangement, as follows: **5**

$$\frac{\mathbf{M}}{\mu\Omega\mathbf{R}^3} = \mathbf{fcn}(\boldsymbol{\theta}); \quad \text{if } M \propto \theta, \quad \text{then } \frac{\mathbf{M}}{\mu\Omega\theta\mathbf{R}^3} = \mathbf{constant} \quad \textit{Ans.}$$

6 **5** **4**

3) The power P generated by a certain windmill design depends upon its diameter D, the air density ρ, the wind velocity V, the rotation rate Ω, and the number of blades n. (a) Write this relationship in dimensionless form. A model windmill, of diameter 50 cm, develops 2.7 kW at sea level when V = 40 m/s and when rotating at 4800 rev/min. (b) What power will be developed by a geometrically and dynamically similar prototype, of diameter 5 m, in winds of 12 m/s at 2000 m standard altitude? (c) What is the appropriate rotation rate of the prototype?

(30)

Solution: (a) For the function $P = \text{fcn}(D, \rho, V, \Omega, n)$ the appropriate dimensions are $\{P\} = \{ML^2T^{-3}\}$, $\{D\} = \{L\}$, $\{\rho\} = \{ML^{-3}\}$, $\{V\} = \{L/T\}$, $\{\Omega\} = \{T^{-1}\}$, and $\{n\} = \{1\}$. Using (D, ρ, V) as repeating variables, we obtain the desired dimensionless function: 5

$$\frac{P}{\rho D^2 V^3} = fcn\left(\frac{\Omega D}{V}, n\right) \quad \textit{Ans. (a)}$$

6 6 5

(b) At 2000 m altitude, $\rho = 1.0067$ kg/m^3. At sea level, $\rho = 1.2255$ kg/m^3. Since $\Omega D/V$ and n are the same, it follows that the power coefficients equal for model and prototype:

$$\frac{P}{\rho D^2 V^3} = \frac{2700W}{(1.2255)(0.5)^2(40)^3} = \frac{P_{proto}}{(1.0067)(5)^2(12)^3}, \quad 3$$

$$solve \quad \mathbf{P_{proto}} = 5990\ W \approx \mathbf{6\ kW} \quad \textit{Ans. (b)}$$

1

(c) "Geometrically similar" means that n is the same for both windmills. For "dynamic similarity," the advance ratio ($\Omega D/V$) must be the same:

$$\left(\frac{\Omega D}{V}\right)_{model} = \frac{(4800\ r/\text{min})(0.5\ m)}{(40\ m/s)} = 1.0 = \left(\frac{\Omega D}{V}\right)_{proto} = \frac{\Omega_{proto}(5\ m)}{12\ m/s}, \quad 3$$

$$\text{or:} \quad \Omega_{proto} = \mathbf{144\ \frac{rev}{min}} \quad \textit{Ans. (c)} \quad 1$$