**4.7** Consider a sphere of radius $R$ immersed in a uniform stream $U_0$, as shown in Fig. P4.7. According to the theory of Chap. 8, the fluid velocity along streamline $AB$ is given by

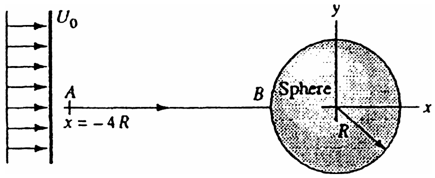

Fig. P4.7

$$\mathbf{V} = u\mathbf{i} = U_0\left(1+\frac{R^3}{x^3}\right)\mathbf{i}$$

Find (a) the position of maximum fluid acceleration along $AB$ and (b) the time required for a fluid particle to travel from $A$ to $B$.

**Solution:** (a) Along this streamline, the fluid acceleration is one-dimensional:

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} = U_o(1+R^3/x^3)(-3U_oR^3/x^4) = -3U_oR^3(x^{-4}+R^3x^{-7}) \quad \text{for } x \le -R$$

The maximum occurs where $d(a_x)/dx = 0$, or at $x = -(7R^3/4)^{1/3} \approx \mathbf{-1.205R}$ *Ans.* (a)
(b) The time required to move along this path from A to B is computed from

$$u = \frac{dx}{dt} = U_o(1+R^3/x^3), \quad \text{or:} \quad \int_{-4R}^{-R}\frac{dx}{1+R^3/x^3} = \int_0^t U_o dt,$$

$$\text{or:} \quad U_o t = \left[x - \frac{R}{6}\ln\frac{(x+R)^2}{x^2-Rx+R^2} + \frac{R}{\sqrt{3}}\tan^{-1}\left(\frac{2x-R}{R\sqrt{3}}\right)\right]_{-4R}^{-R} = \infty$$

It takes **an infinite time** to actually *reach* the stagnation point, where the velocity is zero. *Ans.* (b)

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**4.17** A reasonable approximation for the two-dimensional incompressible laminar boundary layer on the flat surface in Fig. P4.17 is

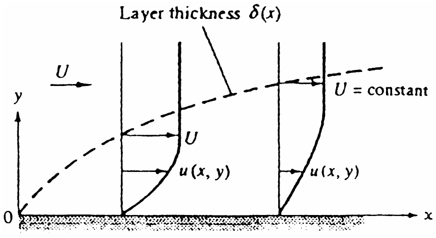

Fig. P4.17

$$u = U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \quad \text{for } y \le \delta$$

$$\text{where} \quad \delta \approx Cx^{1/2}, \ C = \text{const}$$

(a) Assuming a no-slip condition at the wall, find an expression for the velocity component $v(x, y)$ for $y \le \delta$. (b) Then find the maximum value of $v$ at the station $x = 1$ m, for the particular case of airflow, when $U = 3$ m/s and $\delta = 1.1$ cm.

**Solution:** The two-dimensional incompressible continuity equation yields

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -U\left(\frac{-2y}{\delta^2}\frac{d\delta}{dx} + \frac{2y^2}{\delta^3}\frac{d\delta}{dx}\right), \quad \textit{or:} \quad v = 2U\frac{d\delta}{dx}\int_0^y\left(\frac{y}{\delta^2} - \frac{y^2}{\delta^3}\right)dy\Big|_{x=const}$$

$$\textit{or:} \quad \boldsymbol{v = 2U\frac{d\delta}{dx}\left(\frac{y^2}{2\delta^2} - \frac{y^3}{3\delta^3}\right)}, \quad \textit{where} \quad \boldsymbol{\frac{d\delta}{dx} = \frac{C}{2\sqrt{x}} = \frac{\delta}{2x}} \quad \textit{Ans.}\ \text{(a)}$$

(b) We see that $v$ increases monotonically with y, thus $v_{max}$ occurs at $y = \delta$:

$$v_{max} = v\big|_{y=\delta} = \frac{\mathbf{U\delta}}{\mathbf{6x}} = \frac{(3\ m/s)(0.011\ m)}{6(1\ m)} = \mathbf{0.0055\ \frac{m}{s}} \quad \textit{Ans.}\ \text{(b)}$$

This estimate is within 4% of the exact $v_{\max}$ computed from boundary layer theory.

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**4.36** A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle *θ*, as in Fig. P4.36. The velocity profile is

$$u = Cy(2h - y) \qquad v = w = 0$$

Find the constant *C* in terms of the specific weight and viscosity and the angle *θ*. Find the volume flux *Q* per unit width in terms of these parameters.

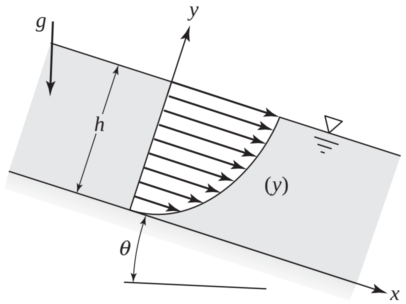

**Fig. P4.36**

**Solution:** There is atmospheric pressure all along the surface at y = h, hence $\partial p/\partial x = 0$. The x-momentum equation can easily be evaluated from the known velocity profile:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\nabla^2 u, \quad \text{or:} \quad 0 = 0 + \rho g \sin\theta + \mu(-2C)$$

$$\text{Solve for} \quad \mathbf{C = \frac{\rho g \sin\theta}{2\mu}} \quad \textit{Ans.}\ \text{(a)}$$

The flow rate per unit width is found by integrating the velocity profile and using C:

$$Q = \int_0^h u\,dy = \int_0^h Cy(2h - y)\,dy = \frac{2}{3}Ch^3 = \mathbf{\frac{\rho g h^3 \sin\theta}{3\mu}} \text{ per unit width} \quad \textit{Ans.}\ \text{(b)}$$

---

**4.41** As mentioned in Sec. 4.10, the velocity profile for laminar flow between two plates, as in Fig. P4.40, is

$$u = \frac{4u_{\max}y(h-y)}{h^2} \qquad \upsilon = w = 0$$

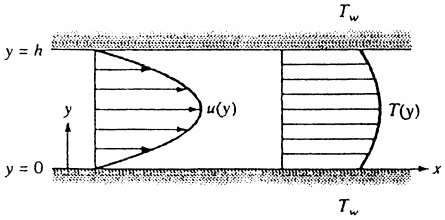

**Fig. P4.41**

the incompressible-flow energy equation (4.75) to solve for the temperature distribution *T*(*y*) between the walls for steady flow.

**Solution:** Assume T = T(y) and use the energy equation with the known u(y):

$$\rho c_p \frac{DT}{dt} = k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2, \quad \text{or:} \quad \rho c_p(0) = k\frac{d^2T}{dy^2} + \mu\left[\frac{4u_{\max}}{h^2}(h - 2y)\right]^2, \quad \text{or:}$$

$$\frac{d^2T}{dy^2} = -\frac{16\mu u_{\max}^2}{kh^4}(h^2 - 4hy + 4y^2), \quad \text{Integrate:} \quad \frac{dT}{dy} = \frac{-16\mu u_{\max}^2}{kh^4}\left(h^2y - 2hy^2 + \frac{4y^3}{3} + C_1\right)$$

Before integrating again, note that dT/dy = 0 at y = h/2 (the symmetry condition), so $C_1 = -h^3/6$. Now integrate once more:

$$T = -\frac{16\mu u_{max}^2}{kh^4}\left(h^2\frac{y^2}{2} - 2h\frac{y^3}{3} + \frac{y^4}{3} + C_1 y\right) + C_2$$

If $T = T_W$ at y = 0 and at y = h, then $C_2 = T_W$. The final solution is:

$$\mathbf{T = T_w + \frac{8\mu u_{max}^2}{k}\left[\frac{y}{3h} - \frac{y^2}{h^2} + \frac{4y^3}{3h^3} - \frac{2y^4}{3h^4}\right]} \quad \textit{Ans.}$$

---

**4.79** Study the combined effect of the two viscous flows in Fig. 4.16. That is, find *u*(*y*) when the upper plate moves at speed *V* and there is also a constant pressure gradient (*dp/dx*). Is superposition possible? If so, explain why. Plot representative velocity profiles for (a) zero, (b) positive, and (c) negative pressure gradients for the same upper-wall speed *V*.

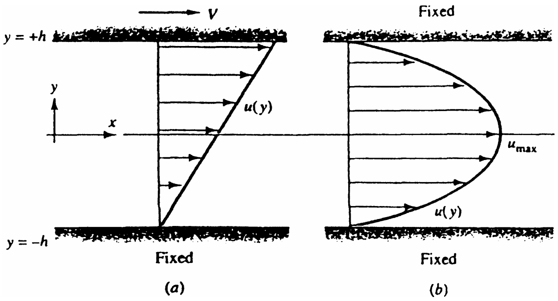

Fig. 4.16

**Solution:** The combined solution is

$$u = \frac{V}{2}\left(1 + \frac{y}{h}\right) + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(1 - \frac{y^2}{h^2}\right)$$

The superposition is quite valid because the convective acceleration is zero, hence what remains is linear: $\nabla p = \mu\nabla^2\mathbf{V}$. Three representative velocity profiles are plotted at right for various (dp/dx).

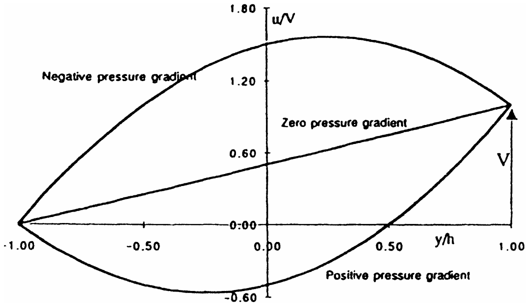

Fig. P4.79

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**4.84** Consider a viscous film of liquid draining uniformly down the side of a vertical rod of radius *a*, as in Fig. P4.84. At some distance down the rod the film will approach a terminal or *fully developed* draining flow of constant outer radius *b*, with $\upsilon_z = \upsilon_z(r)$, $\upsilon_\theta = \upsilon_r = 0$. Assume that the atmosphere offers no shear resistance to the film motion. Derive a differential equation for $\upsilon_z$, state the proper boundary conditions, and solve for the film velocity distribution. How does the film radius *b* relate to the total film volume flow rate *Q*?

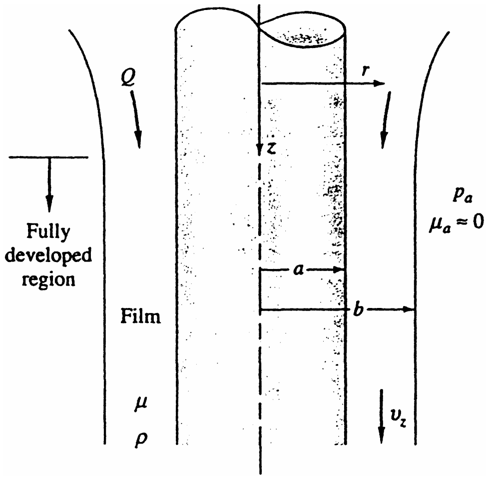

Fig. P4.84

**Solution:** With vz = fcn(r) only, the Navier-Stokes *z*-momentum relation is

$$\rho\frac{dv_z}{dt} = 0 = -\frac{\partial p}{\partial z} + \rho g + \mu\nabla^2 v_z,$$

$$\text{or:}\quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{\rho g}{\mu},\quad \text{Integrate twice:}\quad v_z = -\frac{\rho g r^2}{4\mu} + C_1\ln(r) + C_2$$

$$\text{The proper B.C. are: } u(a) = 0 \text{ (no-slip) and } \mu\frac{\partial v_z}{\partial r}(b) = 0 \text{ (no free-surface shear stress)}$$

$$\text{The final solution is}\quad \mathbf{v_z = \frac{\rho g b^2}{2\mu}\ln\left(\frac{r}{a}\right) - \frac{\rho g}{4\mu}(r^2 - a^2)}\quad \textit{Ans.}$$

$$\text{The flow rate is}\quad Q = \int_a^b v_z 2\pi r\,dr = \boldsymbol{\frac{\pi\rho g a^4}{8\mu}(-3\sigma^4 - 1 + 4\sigma^2 + 4\sigma^4\ln\sigma),}$$

$$\text{where}\quad \sigma = \frac{b}{a}\quad \textit{Ans.}$$

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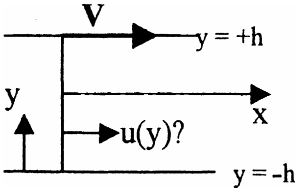
**4.91** Consider 2-D incompressible steady Couette flow between parallel plates with the upper plate moving at speed *V*, as in Fig. 4.16a. Let the fluid be *nonnewtonian*, with stress given by



$$\tau_{xx} = a\left(\frac{\partial u}{\partial x}\right)^c \quad \tau_{yy} = a\left(\frac{\partial v}{\partial y}\right)^c \quad \tau_{xy} = \tau_{yx} = \frac{a}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^c, \quad a \text{ and } c \text{ are constants}$$

Make all the same assumptions as in the derivation of Eq. (4.140). (a) Find the velocity profile *u*(*y*). (b) How does the velocity profile for this case compare to that of a newtonian fluid?

**Solution:** (a) Neglect gravity and pressure gradient. If u = u(y) and v = 0 at both walls, then continuity specifies that v = 0 everywhere. Start with the *x*-momentum equation:

$$\rho\left(u\cancel{\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}}\right) = \rho \cancel{g_x} - \cancel{\frac{\partial p}{\partial x}} + \cancel{\frac{\partial \tau_{xx}}{\partial x}} + \frac{\partial \tau_{xy}}{\partial y}$$

Many terms drop out because v = 0 and $\tau_{xx}$ and $\partial u/\partial x = 0$ (because *u* does not vary with *x*). Thus we only have

$$\frac{\partial \tau_{xy}}{\partial y} = \frac{d}{dy}\left[\frac{a}{2}\left(\frac{du}{dy}\right)^c\right] = 0,\quad or:\quad \frac{du}{dy} = constant,\quad u = C_1 y + C_2$$

The boundary conditions are no-slip at both walls:

$$u(y = -h) = 0 = C_1(-h) + C_2;\quad u(y = +h) = V = C_1(+h) + C_2,\quad solve\ \ C_1 = \frac{V}{2h},\ C_2 = \frac{V}{2}$$

The final solution for the velocity profile is:

$$\boldsymbol{u(y) = \frac{V}{2h}y + \frac{V}{2}}\quad \textit{Ans.}\text{ (a)}$$

This is **exactly the same** as Eq. (4.140) for the newtonian fluid! *Ans.* (b)

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**P4.94** A long solid cylinder rotates steadily in a very viscous fluid, as in Fig. P4.xx. Assuming laminar flow, solve the Navier-Stokes equation in polar coordinates to determine the resulting velocity distribution. The fluid is at rest far from the cylinder. [HINT: the cylinder does not induce any radial motion.]

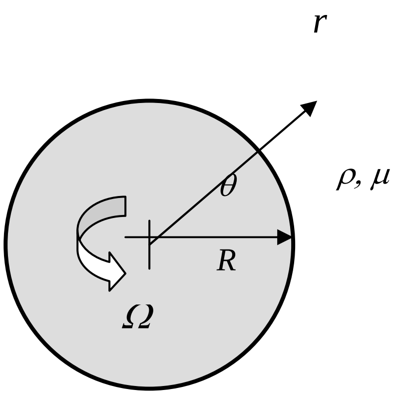


Fig. P4.94

*Solution*: We already have the useful hint that $v_r = 0$. Continuity then tells us that $(1/r)\partial v_\theta/\partial\theta = 0$, hence $v_\theta$ does not vary with $\theta$. Navier-Stokes then yields the flow. From Eq. D.6, the tangential momentum relation, with $\partial p/\partial\theta = 0$ and $v_\theta = f(r)$, we obtain Eq. (4.139):

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right) = \frac{v_\theta}{r^2}, \quad \text{Solution:} \quad v_\theta = C_1 r + \frac{C_2}{r}$$

$$\text{As } r \to \infty,\ v_\theta \to 0, \quad \text{hence } C_1 = 0$$

$$\text{At } r = R,\ v_\theta = \Omega R = \frac{C_2}{R};\ C_2 = \Omega R^2;\ \text{Finally,}\ v_\theta = \frac{\Omega R^2}{r} \qquad \textit{Ans.}$$

Rotating a cylinder in a large expanse of fluid sets up (eventually) a *potential vortex flow*.

***P4.95** Two immiscible liquids of equal thickness $h$ are being sheared between a fixed and a moving plate, as in Fig. P4.95. Gravity is neglected, and there is no variation with $x$. Find an expression for (*a*) the velocity at the interface; and (*b*) the shear stress in each fluid. Assume steady laminar flow.

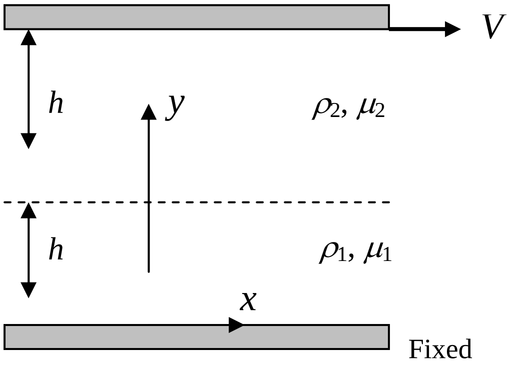


Fig. P4.95

*Solution*: Treat this as a *Ch. 4 problem* (not Ch. 1), use continuity and Navier-Stokes:

$$\text{Continuity}: \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 + \frac{\partial v}{\partial y} = 0 \;;\quad \text{thus}\;\; v = const = 0 \text{ for no} - \text{slip at the walls}$$

This tells us that there is no velocity $v$, hence we need only consider $u(y)$ in Navier-Stokes:

$$\rho_{1,2}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu_{1,2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{or}: \; 0 + 0 = 0 + \mu_{1,2}\left(0 + \frac{d^2 u}{dy^2}\right)$$

$$\text{Thus} \quad u = a + b\,y$$

The velocity profiles are linear in $y$ but have a different slope in each layer. Let $u_I$ be the velocity at the interface. (*a*) The shear stress is the same in each layer:

$$\tau = \mu_1 \frac{u_I}{h} = \mu_2 \frac{V - u_I}{h} \quad \text{Solve for} \quad u_I = \frac{\mu_2}{\mu_1 + \mu_2} V \qquad \textit{Ans.}(a)$$

(*b*) In terms of the upper plate velocity, $V$, the shear stress is

$$\tau = \left(\frac{\mu_1 \mu_2}{\mu_1 + \mu_2}\right)\frac{V}{h} \qquad \textit{Ans.}(b)$$

**C4.2** A belt moves upward at velocity V, dragging a film of viscous liquid of thickness *h*, as in Fig. C4.2. Near the belt, the film moves upward due to no-slip. At its outer edge, the film moves downward due to gravity. Assuming that the only non-zero velocity is *v*(*x*), with zero shear stress at the outer film edge, derive a formula for (a) *v*(*x*); (b) the average velocity $V_{\text{avg}}$ in the film; and (c) the wall velocity $V_{\text{C}}$ for which there is no net flow either up or down.

(d) Sketch *v*(*x*) for case (c).

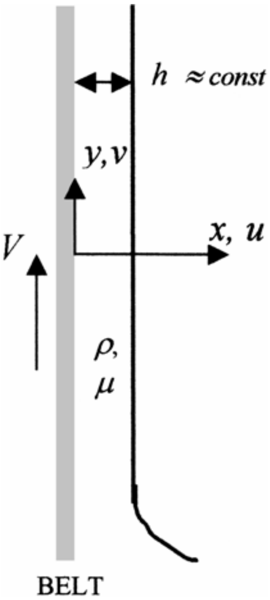


**Fig. C4.2**

**Solution:** (a) The assumption of parallel flow, $u = w = 0$ and $v = v(x)$, satisfies continuity and makes the *x*- and *z*-momentum equations irrelevant. We are left with the *y*-momentum equation:

$$\rho\left(\cancel{u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}}\right)=-\cancel{\frac{\partial v}{\partial y}}-\rho g+\mu\left(\frac{\partial^2 v}{\partial x^2}+\cancel{\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}}\right)$$

There is no convective acceleration, and the pressure gradient is negligible due to the free surface. We are left with a second-order linear differential equation for *v*(*x*):

$$\frac{d^2v}{dx^2}=\frac{\rho g}{\mu} \quad \text{Integrate: } \frac{dv}{dx}=\frac{\rho g}{\mu}x+C_1 \quad \text{Integrate again: } v=\frac{\rho g}{\mu}\frac{x^2}{2}+C_1x+C_2$$

At the free surface, $x = h$, $\tau = \mu(\mathrm{d}v/\mathrm{d}x) = 0$, hence $\mathrm{C}_1 = -\rho gh/\mu$. At the wall, $v = V = \mathrm{C}$ . The solution is

$$\boldsymbol{v = V - \frac{\rho g h}{\mu}x + \frac{\rho g}{2\mu}x^2} \quad \textit{Ans.}\text{ (a)}$$

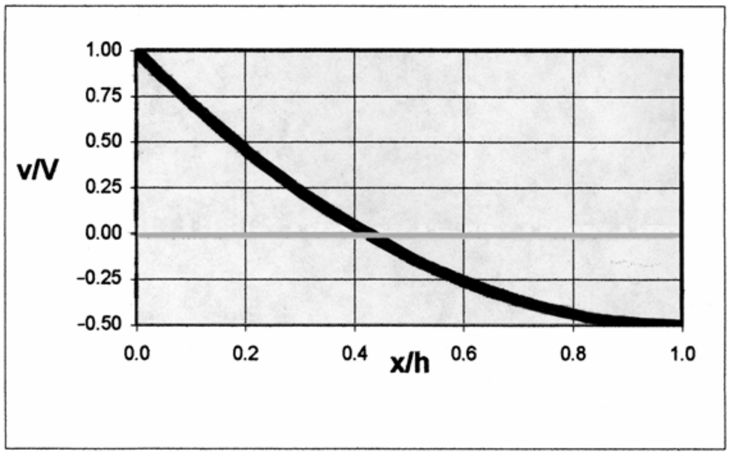
(b) The average velocity is found by integrating the distribution *v*(*x*) across the film:

$$v_{avg}=\frac{1}{h}\int_0^h v(x)\,dx=\frac{1}{h}\left[Vx-\frac{\rho g h x^2}{2\mu}+\frac{\rho g x^3}{6\mu}\right]_0^h=\boldsymbol{V-\frac{\rho g h^2}{3\mu}} \quad \textit{Ans.}\text{ (b)}$$

(c) Since $hv_{\text{avg}} \equiv Q$ per unit depth into the paper, there is no net up-or-down flow when

$$\boldsymbol{V = \rho g h^2/(3\mu)} \quad \textit{Ans.}\text{ (c)}$$

(d) A graph of case (c) is shown below. *Ans.* (d)



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